J. Fluid Mech. (1968), *vol.* 31, *part* 4, *pp.* 689–703

Printed in Great Britain

Report on the International Conference on stratified fluids

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(Received 20 July 1967)

A conference on 'Stratified Fluids' was held in Ann Arbor, Michigan, 11–14 April 1967, under the sponsorship of the National Science Foundation. The meeting was organized by Professor C.-S. Yih (University of Michigan) with the assistance of the other members of the scientific committee: T. B. Benjamin, W. R. Debler, L. N. Howard, R. R. Long, J. W. Miles, W. H. Munk and O. M. Phillips. Thirty papers were delivered on subjects involving the application of stratified fluid problems to geophysical phenomena, waves in stratified fluids, experimental and observational investigations, and stability. We give here a brief account of the proceedings. The papers delivered will not be published in a formal volume; references to where they can be found are given at the end of this article.

1. Introduction

There has been a great surge of interest in geophysical fluid mechanics in the past decade. This has led to a symposium on rotating fluid systems held at La Jolla, California, in the spring of 1966 (Bretherton, Carrier & Longuet-Higgins 1966), and now to a symposium on stratified fluid systems. These two subjects are, of course, fundamental to geophysics because of the rotation of the earth and the ever-present stratification of density or potential density in the oceans and atmosphere.

Papers on stratified-fluid systems appear here and there in the older literature of the nineteenth and twentieth centuries. Among the earliest are papers by Stokes (1847) on waves in a system of two superimposed fluids, and by Burnside (1889) and Love (1891) on waves in continuously stratified fluids. The interplay of shear and stratification in the problem of stability of stratified flows dates back to Helmholtz (1868). Other contributions of importance include the work on stability by Taylor (1931), the circulation theorem of Bjerknes (1898), and the motion of obstacles in stratified fluids (Küttner 1938; Lyra 1943). Problems in which both rotation and statification are important were studied by many people interested primarily in the meteorological and oceanographic applications rather than the basic fluid mechanics of the problems. These included the contributions of Bjerknes *et al.* (1933) on the problem of the stability of discontinuity surfaces in rotating, stratified fluids, the potential vorticity theorem of Ertel (1942) and the problem of inertial oscillations in rotating, stratified fluids by Rossby (1938).

During and immediately after the Second World War, research proceeded slowly although many of the basic studies of the past decade had their origins in this period. Examples of these are: oscillations of stratified fluids by Görtler (1943), long-wave phenomena in stratified fluids by Keulegan (1953) and Long (1956), stability of rotating and stratified fluids by Charney (1947) and Eady (1949), experimental work in rotating stratified systems by Fultz (1953) and Hide (1953), and in stratified liquids by Long (1955), and mountain flows by Scorer (1949).

Recent work, including that reported at this conference, represents a remarkable increase in effort to extend earlier investigations and to initiate new lines of investigation. We must acknowledge that there have been no fundamental breakthroughs in this area to be compared, say, with Prandtl's boundary layer theory, but much recent work and many of the papers reviewed below yield valuable extensions of older work on homogeneous and stratified fluids.

2. Wakes in stratified fluids

The problem of the motion of obstacles in stratified fluids received attention in three papers delivered at the conference. An early investigation along these lines was made by Long (1959), who found a similarity solution for the motion far upstream of an obstacle. The motion consists of a blocking of the fluid at the obstacle level and jets above and below this level. Similar effects occur when obstacles move along the axis of rotation of a rotating fluid.† G. S. Janowitz* presented a general attack on this problem. The slow-motion solution he presented was valid at large distances from the body so that the details of the body shape could be ignored and the flow considered as induced by a point disturbance having a finite drag. His upstream solutions reduced to those of Long at sufficiently great distances, and he resolved the paradox that similarity solutions of the boundary-layer equations do not exist downstream. He found decaying waves downstream which are not permitted by the boundary approximation. His theoretical approach is similar to that of Childress (1964) in his investigation of motion of a body in a rotating fluid.

The problem of motion in the vicinity of an obstacle was investigated in a special case in the paper by R. R. Long & S. Martin.* Here a similarity solution neglecting diffusion and inertial forces was obtained for slow stratified flow over a flat plate. An experiment involving a plate moving slowly through a channel of linearly stratified salt water verified the theory for a narrow range of the parameters. The outstanding feature was the decrease of thickness of the plate boundary layer with distance downstream. The singularity of the solution occurred at the back of the plate.

† A recent paper by Bretherton (1967) shows that certain aspects of blocking may be explained in terms of internal wave propagation in a perfect fluid.

* Indicates paper presented at the conference.

Another aspect of wakes in stratified fluids was discussed by M. Tulin.* In part, his paper was concerned with the collapse of a turbulent wake behind a body moving in a stratified medium. When such wakes occur or, for example, when turbulence accompanies the breaking of an internal wave, the mixed region tends to collapse and spread horizontally. The collapse is accompanied by the generation and propagation of internal waves. The problem has importance when related to the motion of submarines in or near the thermocline. The body itself does not create internal waves of any importance, but the wake has a collapse period close to that of internal wave periods and, therefore, generates waves very efficiently. Tulin's experimental investigations, presented on film, showed the generation of internal waves by a pulsating source of disturbance, and also quasi-steady propagation of a homogeneous body of fluid.

3. Observation and experiment

An experimental paper by W. Debler* involved a joint problem of convection and stratification. Debler generated a heated plume in a wind tunnel in which there was a basic linear, vertical temperature gradient in the air moving over the heated strip. The plume was identified by temperature measurements on a movable probe downstream of the heat source. Measurements indicated a high degree of uniformity of turbulent mixing in the plume.

Three papers by J. Crease,* G. Dowling* and T. Pochapsky* were concerned with observations of stratified fluid phenomena in the oceans. The paper by Crease represented an attempt to study the relationship between stability, flow profile and vertical mixing in the Faeroe Bank Channel between the Atlantic Ocean and the Norwegian Sea. Observations of interest were a strong thermocline near the bottom of the channel and an associated deep-water current with speeds up to 15 cm/sec.

The paper by Dowling described an investigation of the internal waves in shallow water having a strong stratification during the summer months. From the cross-spectral analyses of isotherm fluctuations, low frequency, internal tides were observed which follow the surface tides. The data were found to satisfy the characteristic equation found from internal wave theory if the density is assumed to vary as the hyperbolic tangent with depth. This theory enables one to determine the variation of internal wave phase velocity with depth at any given frequency. The effect of interaction between the waves and the shear flow is that when the shear is greater than the phase velocity, the top part of the internal wave is sheared off, causing the wave energy to be dissipated in the upper and lower layers by turbulent mixing. This observation accounts in part for the low coherence of internal waves as a result of the co-existence of internal waves and turbulence. The theory and experimental data seem to support the concept of interaction of internal waves, shear flows, and turbulence, giving rise to an energy transfer from one part of an internal wave spectrum to another.

T. E. Pochapsky presented some measurements of internal waves and turbulent motions in the deep ocean well away from land. Conditions were sought in which the Richardson number Ri is of order one or greater, since, if Ri is much smaller, turbulent mixing would be lacking and that part of the ocean would be virtually stagnant for a long time. Neutrally buoyant floats, equipped with pressure and temperature sensing devices, were dropped in clusters.

D. Harleman & J. Dake* discussed the problem of the generation and maintenance of thermal stratification in quiescent lakes by incoming solar radiation. Although theoretical in nature, this work was inspired by observations of the temperature fields in lakes such as Tahoe. The authors made objection to earlier work on the subject which assumed that the heat was absorbed completely at the surface, on the grounds that very large coefficients of turbulent diffusion have to be postulated for such models to explain existing observations. The present paper developed a theory that fits observations fairly well by neglecting fluid motions and assuming that much of the incoming radiation is absorbed at various depths. In discussion of the paper, objection was raised on the grounds that the fluid motion is of importance. Motions at great depths can certainly be generated by internal waves and the breaking of these can create turbulence which could easily overshadow the molecular processes envisioned by Harleman & Dake. Indeed there is evidence of considerable turbulence at great depths in the oceans.

4. Interplay of rotation and stratification

Three papers by V. Barcilon & J. Pedlosky,* G. Veronis* and G. Walin,* considered the interplay of rotation and stratification. The subject is of great importance to meteorology and oceanography because large-scale phenomena in atmosphere and oceans are dominated by the effects of both of these. These papers together with earlier work dating back to Ekman, make the properties of boundary layers in these systems reasonally clear. With weak or moderately strong stratification, horizontal boundaries possess an Ekman layer in which viscous and Coriolis forces are in essential balance. The thickness of this layer is

$$L = (\nu/\Omega)^{\frac{1}{2}}$$

where ν is the viscosity and Ω is the angular velocity. The boundary-layer structure on vertical walls is more complicated. With zero stratification, this layer, called the Stewartson layer, has a double structure. An inner layer has a thickness of the order of $E^{\frac{1}{3}}$ where E is the Ekman number, and an outer layer a thickness of the order of $E^{\frac{1}{4}}$. The presence of a moderate amount of stratification has no effect on the Ekman layer, but the $E^{\frac{1}{3}}$ side-wall layer splits into a double layer called a buoyancy and a hydrostatic layer. The outer Stewartson layer is still present. Barcilon & Pedlosky presented a study of these boundary layers in a fluid cylinder composed of rotating outer walls and a top and bottom rotating differentially. A constant stratification is imposed and, since there is no heat transfer through the walls, attention is confined to periods of time before any substantial tendency toward uniform density takes place. In addition to the Ekman number, E, the phenomena depend on a number σS where σ is the Prandtl number and $S^{\frac{1}{2}}$ is the ratio of the Brunt–Väisälä frequency to the angular velocity. It was shown that the theory of homogeneous fluids applies when $\sigma S < E^{\frac{2}{3}}$, and attention was confined to the region of intermediate stratification $E^{\frac{2}{3}} < \sigma S < E^{\frac{1}{2}}$. It is in this intermediate region that the triple structure of the side-wall boundary layer occurs. With moderate stratification, the interior motion is mainly controlled by Ekman layer suction. The motion can be decomposed into two components: the first satisfies the Taylor–Proudman theorem of a homogeneous fluid, and the second, a baroclinic component, satisfies the thermal wind relationship.

Veronis's paper was concerned with the analogy between rotating and stratified fluids, but its results have strong bearing on the boundary-layer problems considered by Barcilon & Pedlosky. It has been known for a long time that there is a close relationship between stratified and rotating fluids. Occasionally one can demonstrate an exact correspondence. A famous example is provided by the stability problems of a non-rotating fluid between two horizontal heated plates and the homogeneous fluid contained between two concentric rotating cylinders with a narrow gap. Veronis showed that there is an exact correspondence of the differential equations of linearized, steady, two-dimensional (axisymmetric) rotating and stratified cases. The total problems are exactly analogous for closed containers of fluid when vorticities are prescribed at the top and bottom boundaries in one case, and temperatures at the lateral boundaries in the other case. These have been called the spin-up and heat-up problems, respectively. Corresponding quantities in the rotating and stratified cases are $z \sim x$, $u \sim w$, $w \sim u$, $v \sim T$, $E \sim R$ where R is the Rayleigh number. The analogy breaks down for time-dependent and three-dimensional problems. Veronis also showed how one can arrive at the rotating and stratified fluid problems by adding stratification to the rotating case and rotation to the stratified case. The stratified problem yields an 'Ekman' layer on vertical walls and $E^{\frac{1}{3}}$ and $E^{\frac{1}{4}}$ layers on horizontal walls. The Ekman layer has the familiar spiral structure, although the spiral is in the (w, T)-plane rather than the (u, v)-plane.

The paper by Walin reported an attempt to obtain an integrated understanding of boundary layers in rotating and stratified fluids along surfaces with an arbitrary orientation with respect to the gravitation and rotation vector. The results may be summarized in terms of a parameter B, which is proportional to the product of the Brunt–Väisälä frequency, and the sine of the angle between the gravitation vector and the normal to the boundary. When B is large, for example, one of the boundary layers becomes formally identical with the Ekman layer but, as predicted by Veronis's paper, the density or temperature field replaces one of the velocity components in the Ekman layer.

5. Stability of stratified flows

A session on stability began with an introductory discussion by L. N. Howard* on stability criteria for stratified flows. Howard discussed three general stability criteria for parallel flows. For instability it is necessary that

(i) $D^2w = 2(w-c_r)J(y)/[(w-c_r)^2+c_i^2]$ somewhere in the flow field. This is Rayleigh's criterion for stratified flows.

(ii) $[c_r - \frac{1}{2}(w_{\min} + w_{\max})]^2 + c_i^2 < [\frac{1}{2}(w_{\max} - w_{\min})]^2$. This is due to Howard.

(iii) $J < \frac{1}{4}$. This result is due to J. W. Miles.

Here $J(y)$ is the Richardson number, w is the primary velocity, and $c = c_r + ic_i$ is the complex wave speed. All of these criteria can be obtained from the stability equation by considering the basic variable to be $(w-c)^{n-1}$ times the streamfunction, and integrating the multiplied equation in the domain of interest. For theorems (i), (ii) and (iii), n takes on the values 1, 0 and $\frac{1}{2}$, respectively. Howard's attempts to generalize these to non-parallel flows were successful only in the case of no stratification, for which the flow was found to be stable if the basic vorticity decreases toward the right everywhere looking downstream along the streamlines. Howard pointed out that the lack of success in further generalization of these criteria to non-parallel, stratified flows is probably related to an insufficient understanding of the physical relevance of the transformed streamfunction.

A paper by J. Miles* considered the particular case when the velocity and the logarithm of the density vary exponentially with height. When the Boussinesq approximation was made, the solution could be found in terms of a hypergeometric function, which predicted stability for all wavelengths and Richardson numbers. The paper was ably presented by H. Huppert in Miles' absence.

An interesting approach to inviscid stability problems was given by R. S. Scorer,* who considered the stability of spiral flows in the inviscid limit. Introducing a local instability by rotating the vorticity vector a small amount about the streamline, the displacement is unstable if the rate of change of the vorticity component is in the same direction as the angular displacement of the vorticity vector, that is, the vortex line must contract. The unstable directions are found to be those which lie between the vorticity vector and the direction of the axis of the vortex. In particular, when the disturbed vorticity vector bisects the angle between these two vectors, the growth rate is maximized. Scorer's theory is a generalization of Rayleigh's theorem and shows that toroidal disturbances, as exhibited by Taylor–Görtler cells, are the most unstable.

M. E. Stern* presented some considerations of the effect of a long wave superimposed on a salinity profile varying sinusoidally in the horizontal direction. He speculated that advection of the salt fingers in groups will produce a growing buoyancy force. The local wave energy would then decay until it is sufficiently quiet for the salinity fingers to grow again, restarting the whole process.

The growth of interfacial waves in a long tube filled with a two-layer fluid was demonstrated in films shown by S. A. Thorpe.* A horizontal tube of rectangular cross-section containing stratified fluid was suddenly raised to an inclined position. The pictures then showed a fairly regular array of roll waves forming ahead of the surge. The wave patterns were clearest when the fluids were miscible, i.e. brine and fresh water. The oil and water pictures appeared less well defined.

Viscosity effects on stratified flows were considered in several papers. A layer of viscous fluid in simple shear under an inviscid thick layer moving at uniform velocity was considered by D. Y. Hsieh,* who found that for long waves the presence of the shear flow tended to stabilize the system. Some doubts were expressed as to the reasonableness of considering a primary viscous flow with a discontinuity in shear stress.

The interaction of thermal instabilities and the instabilities associated with shear flows were considered in separate papers by K. W. Gage & W. H. Reid,* A. J. Faller & R. Kaylor* and E. Palm, T. Ellingsen & B. Gjevik.* Gage & Reid, using classical, linear stability theory, considered the basic flow to be a parabolic velocity profile with a linear temperature gradient. They found that when the stratification is hydrostatically unstable and the Prandtl number is unity, one-to-one mathematical correspondence could be found between this flow and spiral flow between concentric cylinders. Using an asymptotic analysis with the Richardson number J held fixed, they found a critical value of J ($J_c = -0{\cdot}366\times10^{-5}$) above which the instability was of the Tollmien–Schlichting type. Below this value thermal instability governs, Bénard cells dominate, and Squire's theorem is no longer valid. Similar effects were found by Faller & Kaylor, who made a numerical investigation of the interaction of a shear flow in an Ekman layer with thermal stratification. In particular, when thermal and shear instabilites were at the point of almost equal growth rate, the roll vortices associated with each mechanism moved with different speeds, interacting so that vorticity was continually exchanged from one horizontal layer to another.

Palm *et al.* presented an expression by which one can determine whether hexagons and/or rolls are present in Bénard convection. When the Rayleigh number exceeds the critical value R_c, and lies between values R_1 and R_2, both hexagons or rolls are stable. For values above R_2, however, rolls are the only stable mode. R_1 and R_2 are given by the expression

$$R_j - R_c = \chi_j\left(\frac{K\nu_0\gamma}{\alpha g h^3}\right)^2 R_c^3 \quad (j = 1,\ 2),$$

where χ_1 and χ_2 are found by numerical methods and are relatively independent of the boundary conditions. From the formula it is seen that to observe hexagons, i.e. to have $R < R_2$, thin layers are essential.

A lecture by D. Fultz & J. Kaiser* summarized the results of the 'dishpan experiments' conducted by both the Chicago group and the MIT group. They pointed out that the difficulty of exact comparison was due to differing methods of measuring horizontal temperature gradients. Stability analyses by Kuo, Barcilon, Brindley and Lorenz pertain to these experiments, the first two giving very good agreement with experiments with respect to neutral stability curves. Kuo's results for wave-numbers agree more closely than Barcilon's with those observed in the experiments. One difficulty with the four theories mentioned is that all of them require the stratification number to be proportional to the square of the Rossby number, whereas a linear relation is more appropriate to most of the experiments.

6. Geophysical phenomena

A survey lecture 'The Ocean' was given by O. M. Phillips.* In a typical situation, the ocean has an upper layer of constant density, below which the density increases with depth. The density stratification is characterized by a seasonal thermocline and a permanent thermocline in addition to other minor variations in the lower layer. The discussion was aimed at the dynamical characteristics of the ocean and some types of interacting motions in which the density stratification plays an important role.

In the simple case of constant Väisälä frequency N, there are internal waves of frequency $n = N \cos\theta$, where θ is the angle between the wave vector $\mathbf{k}$ and the horizontal. The group velocity, $\mathbf{c}_g = \partial n/\partial \mathbf{k}$, is always perpendicular to the phase velocity $\mathbf{c}_p = n\mathbf{k}/\mathbf{k}^2$ and lies in the vertical plane containing $\mathbf{c}_p$ such that the sum of $\mathbf{c}_p$ and $\mathbf{c}_g$ gives a horizontal vector of magnitude N/k, and independent of θ. Since $\mathbf{c}_g$ is also the energy flux vector, various modes of propagation of internal waves can be predicted on this basis. For instance, when $\theta = \frac{1}{2}\pi$, a vertically localized wave packet propagates in its own horizontal layer. Also, a wave group having an inclined energy flux propagating into a region bounded by a free surface above and a sloping ocean bottom below may undergo a multiple reflexion at the free surface and the bottom.

In a more general case of variable Väisälä frequency $N(z)$, the internal waves with frequency n such that $n < N(z)$ for $z_1 < z < z_2$ and $n > N(z)$ outside this region are clearly trapped in the layer $z_1 < z < z_2$, a phenomenon known as the Väisälä trapping.

The non-linear effects may give rise to resonant interaction between a set of internal waves; energy can be exchanged between these waves when their wave-numbers $\mathbf{k}_1$, $\mathbf{k}_2$, $\mathbf{k}_3$ and frequencies n_1, n_2, n_3 satisfy the conditions

$$\mathbf{k}_1 \pm \mathbf{k}_2 = \mathbf{k}_3, \quad n_1 \pm n_2 = n_3,$$

The amplitude of the wave generated by this interaction remains small if N is large compared with the rates of shear (or vorticity) in the wave modes (Phillips 1966). However, if the vorticity is large, the forced wave may be strong.

Another interesting phenomenon is the interaction of internal waves with a shear flow. The effect of a uniform weak shear in the horizontal direction on a train of internal waves is to change their wave-number $\mathbf{k}$ to be directed more toward the vertical, thereby shifting the energy spectrum to a lower frequency band, until $\mathbf{k}$ is vertical and frequency vanishes. With a slight generalization, a slowly varying, horizontal weak shear can be considered as the limit of an internal wave having zero frequency and $\mathbf{k}$ vertical. This wave mode acts as a catalyst for the interaction between two internal waves $n_1 = n_2$, by which the total energy content flows back and forth between the two waves, resulting in a horizontal channelling of low frequency internal wave propagation (Phillips 1966). This is another mode of trapping of internal waves. When the shear motion is strong, transfer of momentum, without transfer of mass, may occur in critical layers (Miles 1957, 1959).

Also described by Phillips were the wind-driven currents and their interaction with the ambient stratification. The Red Sea was given as an example in which the current system seems to be driven almost entirely by convective processes. Finally, the combined effects of stratification and rotation were discussed for frequencies n lying in $N > n > \Omega \sin \lambda$, where Ω is the rotation speed of the earth, λ is the latitude.

Lofquist* described a detailed calculation of the stratified flows past a sphere and a doublet. The motivation of his method is to try to explain why, when a doublet is used to represent approximately a sphere in a stratified flow, the pressure and velocity become singular on the axis downstream of the doublet. However, several questions were raised during the discussion: is the present solution valid as the boundary condition is satisfied only at a single point on the sphere? Does the body represented by his solution have a closed surface?

7. Wave theory

The session on wave theory started with the paper by T. Brooke Benjamin* who reviewed recent work on finite internal waves in three categories: (i) phenomena akin to hydraulic jumps and bores; (ii) long waves in fluids of limited total depth; and (iii) long waves in fluids whose total depth greatly exceeds the depth over which the density varies significantly.

In (i), several types of steady stratified flow with practical interest were summarized, and it was emphasized that adequate explanations of them are generally provided by analogy with well-understood principles in open-channel hydraulics. One example that was mentioned is the flow of a heavy fluid over a mound, with a lighter fluid above flowing in the opposite direction, as required by continuity. Another is the intrusion of a wedge of heavy fluid, of density ρ_1, moving with constant velocity U into a mass of lighter fluid, of density ρ_2, along a horizontal or slightly inclined plane. These gravity-driven currents are usually characterized by a steeply rising head wave, behind which a zone of vigorous turbulence arises. By neglecting frictional effects and applying Bernoulli's theorem along the interface between the fluids, von Kármán (1940) determined the slope of the interface at its foremost point on the bottom to be $\pi/3$, and its mean height behind the head to be $H = U^2/[2g(\rho_1-\rho_2)/\rho_2]$. However, Benjamin pointed out that von Kármán's argument leading to this expression for H must be repudiated, since a significant loss of total head always occurs along the interface; but happily the same result is forthcoming by an alternative argument which allows for the essentially dissipative character of the flow. Benjamin also showed that consideration of a momentum balance for these gravity currents explains why the head wave must break on its rearward side, and he discussed some of the properties of the turbulent wake that arise in this way.

The theory of waves of permanent form in category (ii) has been extensively developed during the last decade. For the classical water waves of this kind, Airy's theory, which neglects the effects of vertical accelerations, shows that waves of elevation steepen ahead of their crests as the result of non-linear effects. On the other hand, the linearized surface-wave theory allowing for vertical accelerations shows that the steepest parts of a wave group are dispersed and fanned out. Recognizing that internal waves also feature this opposition between non-linear and frequency-dispersive effects, Benjamin (1966) and others have considered the class of flow régimes in which these two tendencies balance each other, and the properties of internal solitary and cnoidal waves have been evaluated. It appears, however, that the order-of-magnitude relationship between the amplitude a and horizontal length scale λ of these waves is $a\lambda^2 = O(h^2/\beta)$, where h is the depth of the fluid and β^{-1} is the scale height for the density variations. Thus, since βh is usually a very small fraction, such waves need to be extremely long in comparison with h if permanency of form is to be ensured, and so they are difficult to realize under laboratory conditions.

Turning to the final topic of his review, Benjamin reported that this difficulty is not presented by waves of finite amplitude and permanent form in category (iii), a study of which he had recently completed—simultaneously, it had turned out, with an independent investigation by R. E. Davis & A. Acrivos.* A new type of solitary wave was found, together with an allied class of periodic waves, and both have waveforms and other properties radically different from waves in category (ii). In particular, the relationship between amplitude and wavelength is $a\lambda = O(h^2)$, where now h is the depth within which the density variation is confined and not the total depth; and the fact that this relationship is independent of the scale height indicates that these waves are readily realizable in the laboratory. During the discussion period, Davis summarized some experiments he had made on solitary waves in category (iii) and he presented a ciné film which convincingly demonstrated their property of permanence.

C.-S. Yih* described a shallow-water theory for finite-amplitude flows of two (or more) superposed layers of inviscid fluids, each of a constant, but different density. Three categories of problems are considered. In the first, the flow motion is due to an initial displacement of the free surface and of the interface, or to the slow horizontal motion of a vertical piston driving the fluids. In these problems, the method of characteristics is used. There are two pairs of characteristics passing through each point in the (x, t)-plane. By using these characteristics the velocity field and the surface profiles can be calculated stepwise, although the computation is more involved than in the single-layer case on account of the interaction between the free surface waves and the interfacial waves.

The second problem described by Yih concerns nearly horizontal gravity jets of a heavy fluid underneath a lighter one at rest, or of a light fluid above a heavier one at rest.

The third problem dealt with the quasi-one-dimensional 'channel' (or 'nozzle') flow of a two-layer system with a free surface and an interface. The computation was made particularly simple by taking the two layers to be of equal height, although arbitrary heights can be treated in the same manner. We understand that the solutions of the last two problems have now been extended to apply to continuously stratified fluids.

A. Foldvik* described a study of the linearization procedure for two-dimensional, steady, stratified shear flows of an inviscid, non-diffusive fluid. The validity of the linearization was examined by estimating the neglected non-linear terms. The relative importance of these neglected terms was found to depend on the basic density and velocity and some of their derivatives. The conditions for which the linearized vorticity equation yield exact solutions were presented; these conditions are equivalent to those already given by Yih (1960) using a different approach.

T. W. Wu* described the general problem of determining the wave field produced in a stratified fluid by a concentrated disturbance moving with a given velocity **V** in an arbitrary direction, and with its strength oscillating at a fixed frequency ω_0. The discussion covered the simple case of constant Väisälä frequency N and constant **V**, and touched on the general case of arbitrary $N(z)$. In the first case, the steady-state limit of the phase function of the internal waves in an unbounded fluid can be readily obtained from a simple consideration of the principle of stationary phase and the concept of transport of wave energy. These waves occupy a region which extends outwards along the front $\mathbf{x} = \mathbf{c}_{ge}\, t$ ($\mathbf{c}_{ge}$ being the group velocity relative to the disturbance) into the undisturbed region if the motion has started at $t = 0$; the waves in a neighbourhood of the front have a more complicated form, but with their amplitude diminishing with increasing time. For the case of arbitrary $N(z)$, the wave field was evaluated by using the geometric wave approximation. The general results were applied to several specific examples: (i) oscillating dipole of Görtler (1943), (ii) two-dimensional dipole in horizontal motion (Wu & Mei 1967), (iii) horizontally moving three-dimensional dipole (Wu 1965), (iv) rising or sinking dipoles. These solutions exhibit some singular behaviour in certain regions of the wave field, suggesting that either the neglected effects of non-linearity and viscosity may be important there, or the first-order stationary-phase approximation is too crude for these regions.

For the purpose of comparison, the method of Fourier transform was briefly reviewed. The indeterminacy of the steady-state solution, arising from the occurrence of branch points and poles on the integration path of the inverse transform, can be resolved by formulating an initial value problem and then taking the large time asymptotic limit. This difficulty is curtailed in the method of Lighthill (1967) when the Fourier transform is applied to all space co-ordinates and the time.

In the course of discussion, Carrier commented on whether the singularities of the solution can be removed by a more accurate asymptotic calculation. Wu thought that this is very likely the case for the singularity along the course of a point disturbance in horizontal motion. However, it appears not so for the singularity along the boundary of the wave field emitted by the Görtler oscillating dipole, since the exact solution of this linear problem, which can be expressed in terms of the Hankel functions, has a logarithmic singularity, and possibly also poles, at this surface; this would suggest the importance of non-linear and viscous effects in this region.

Another contribution to the study of the phase configuration of internal waves was the paper delivered by B. S. H. Rarity.* Three problems were discussed and the corresponding experimental results were presented. The first was the problem of the internal waves produced by a point disturbance oscillating with a small amplitude about the origin. Determination of the phase configuration of the waves based on the stationary-phase principle and on the group-velocity concept was explained. The theoretical prediction, which is the same as the earlier result of Görtler and others, is in agreement with experiments conducted with a forcing frequency ranging from 0·2 to 1·11 of the Väisälä frequency (slightly inhomogeneous). The second problem was the Cauchy–Poisson type initial-value problem of a point disturbance moving impulsively in a stratified fluid. The asymptotic solution of the wave field has two families of waves, an oblique and a transverse wave, reminiscent of the Kelvin ship waves. The third problem was the motion of a point disturbance with a constant velocity. Rarity also described the law of reflexion of waves from a rigid surface, and showed that the wavelength changes if the surface at the point of wave impingement is not tangential to the horizontal.

The photographs and several, short motion pictures taken of these internal waves by using a Toepler–Schlieren system are probably the most impressive and striking of all visualizations ever recorded in this kind of experimental effort. Some of the results presented by Rarity have henceforth appeared in publication (Mowbray & Rarity 1967).

Another aspect of wave theory, which is concerned with the viscous effects on internal waves, was contributed by M. Yanowitch,* who dealt with the problem of two-dimensional infinitesimal oscillations of an incompressible, stratified fluid occupying the upper half space bounded below by a horizontal plane, whose profile is a moving corrugation. Yanowitch found four regions of different flow behaviour; a boundary layer next to the plate ($0 < z < z_1$), a layer ($z_1 < z < z_2$) in which the solution can be approximated by some inviscid solutions, a third layer ($z_2 < z < z_3$) in which all the terms of the differential equations are of equal importance, and finally a viscosity-dominant layer ($z > z_3$). Yanowitch showed that viscosity produces some downward reflexion of wave energy from the free upper boundary of a stratified fluid. This possibility is particularly interesting in relation to the propagation of long waves in the atmosphere.

The last paper in the session on wave theory was given by M. C. Shen,* who described two physical models for the study of long waves in a stratified, compressible atmosphere. It has been noted that no solitary wave solution can be found in either an incompressible atmosphere of infinite height with its density decreasing exponentially with height, or in a compressible isothermal atmosphere of infinite height. This study is concerned with the effect of the temperature and velocity variation on the existence of long waves in a compressible atmosphere. The first model used by Shen is a compressible atmosphere of finite height having a given distribution of density $\rho_0(z)$ and velocity $u_0(z)$ of the primary flow, and a known barotropic relationship between the pressure p and density ρ.

The second problem presented by Shen was the extension to the case of unsteady waves in a compressible atmosphere of infinite height with arbitrary density $\rho_0(z)$ and cross-wind profile $u_0(z)$, $v_0(z)$.

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